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Product of Proposition–State Structures Preserving Superposition

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The notion of product is introduced in the framework of proposition–state structures. Central roles are played by the invariance of the superposition relation and the maximality property of the states under the interaction of physical systems.

1. INTRODUCTION

The problem of the description of a compound physical system $\tilde{\Sigma} + \Sigma$ in terms of the interacting physical systems Σ and $\tilde{\Sigma}$ has been formulated in the context of the logical approach to quantum mechanics by many authors (Hellwig and Krausser, 1977; Zecca, 1978, 1981*a*; Aerts, 1982, 1984; Pulmannová, 1983, 1985).

If (L, S) and $(\tilde{L}, \tilde{S})$ are the logics and states of Σ and $\tilde{\Sigma}$, respectively, the solution of the problem is that of obtaining the logics and states $(\mathbf{L}, \mathbf{S})$ of $\Sigma + \tilde{\Sigma}$ in the form $\mathbf{L} = L \otimes \tilde{L}$ and $\mathbf{S} = S \otimes \tilde{S}$, with $\otimes$ being a suitable notion of the product of logics and states. The solutions proposed by the mentioned authors contain, besides the limiting case of the product of classical logics and purely quantum logics, also the cases of the product of a classical and an irreducible quantum logic. This last case is relevant in connection with the problem of quantum measurement, where a classical logic is generally associated with the measuring apparatus and an irreducible quantum logic with the physical system (Jauch, 1968; Ludwig, 1973; Sudarshan, 1976). The product of both logics and states provides also a basis for the formulation of reduced dynamics in the context of quantum logics. This could be done by extending in some way to compound systems

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and subsystems the reversible and irreversible dynamics of conventional quantum mechanics [some results in this direction can be found in Gorini and Zecca (1975) and in Zecca (1976, 1992)].

The mentioned schemes of Hellwig and Krausser, Zecca, and Aerts contain implicitly the notion of the product of states only in the limiting cases, since they can be reduced, via standard representation theorems, to the Hilbert or to the phase space model.

Instead, a product at the abstract level of both logics and states has been explicitly given in the schemes of Pulmannová (1983, 1985).

It is the object of this paper to propose another intrinsic definition of product in the framework of the proposition–state structures. This is obtained by coupling states and propositions in an interdependent way under the condition that the physical systems $\Sigma, \tilde{\Sigma}$ preserve their identities when considered as subsystems of the compound physical system $\Sigma + \tilde{\Sigma}$.

The central request on the theory is the assumption of the invariance under physical interaction of the superposition relation and of the maximality property of the states.

The product of logics which is obtained in this way has many properties similar to those of the coupling of logics previously defined by the author (Zecca, 1978, 1981*a*).

2. THE PROPOSITION–STATE STRUCTURE

Definition 1. A proposition–state structure (pss) is a pair (L, S), where L is a logic, namely a complete, orthomodular, atomistic lattice with greatest and least elements $\bigvee L = \mathbf{1}$ and $\bigwedge L = \varnothing$, respectively, and S is a family of maps $s: L \to [0, 1]$, $s(\mathbf{1}) = 1$, such that, denoting $S_1(x) = \{s \in S: s(x) = 1\}$ and $S_0(x) = \{s \in S: s(x) = 0\}$ (and with $\perp$ being the orthocomplementation in L):

A1. $S_1(x) \subset S_1(y) \Leftrightarrow x \leq y,\ x, y \in L.$
A2. $S_1(\mathbf{1}) = S,\ S_1(x) = S_0(x^\perp).$
A3. $S_1(\bigwedge_\alpha x_\alpha) = \bigcap_\alpha S_1(x_\alpha),\ \forall [x_\alpha\} \subset S.$

Under these assumptions the center $C(L)$ of L, namely the set of the elements of L which commute with all the elements of L, is a completely distributive sublattice of L. As usual, the elements of L (propositions) represent classes of equivalent tests on Σ, and S represents the set of the preparing procedures of Σ. The number $s(x)$ ($s \in S$, $x \in L$) is interpreted as the probability of the outcome yes for a test of x when Σ is in the state s. If D is any set of states, we denote by $L(D)$ the dual principal ideal $L(D) = \{x | x \in L: s(x) = 1\ \forall s \in D\}$ (Berzi and Zecca, 1974).

Definition 2 (Varadarajan, 1968; Berzi and Zecca, 1974). The state s is a superposition of the states $D \subset S$ if $L(s) \supset L(D)$.

For every $D \subset S$ the map $D \to \bar{D} = \{s \in S: L(s) \supset L(D)\}$ is a closure operation whose family of closed subsets of S is exactly $\{S_1(x): x \in L\}$ and $S_1(\bigvee_\alpha x_\alpha) = \overline{\bigcup_\alpha S_1(x_\alpha)}$ for every $\{x_\alpha\} \subset L$. Moreover, the following identities hold (Gorini and Zecca, 1975):

$$\bar{D} = \bigcap \{S_1(x): D \subset S_1(x)\} = S_1(\bigwedge L(D))$$

Lemma 1. For every family $\{D_\alpha\}$ of subsets of S:

(i) $L(\bigcap_\alpha D_\alpha) = \bigcap_\alpha L(D_\alpha)$.

(ii) $\bigcap \bar{D}_\alpha = \overline{\bigcap_\alpha D_\alpha}$.

Proof. (i) This holds from the very definition of $L(D)$. (ii) One has $\bigcap_\alpha D_\alpha \subset \bigcap_\alpha \bar{D}_\alpha$ and hence $\overline{\bigcap_\alpha D_\alpha} \subset \bigcap_\alpha \bar{D}_\alpha$, with $\bigcap_\alpha \bar{D}_\alpha$ being closed by the previous considerations and A3. *If* $s \in \bigcap_\alpha \bar{D}_\alpha$, then $L(s) \supset L(D_\alpha)$ for every α, and, by (i), $L(s) \supset \bigcap_\alpha L(D_\alpha) = L(\bigcap_\alpha D_\alpha)$ and hence $s \in \overline{\bigcap_\alpha D_\alpha}$.

Remark 1. Let L be a Boolean lattice. With our assumptions L is orthoisomorphic ($\cong$) to the power set of its atoms: $L \cong P(A(L))$. Let S^* denote the set of the probability measures on L with total mass $= 1$ and S a subset of S^* which coincides with the atomic measures on L or which contains the atomic measures on L and satisfies assumption A3. From the $1-1$ correspondence between the atomic measures and the atoms of L, one can check that, in both cases, (L, S) is a pss.

Remark 2. Let $L(H)$ be the standard logic associated with a separable complex Hilbert space H of dimension ≥ 3, and let S be the set of the maps $s: L \to [0, 1]$ such that $s(H) = 1$, $s(\bigvee_i x_i) = \sum_i s(x_i)$ if $x_i \perp x_k$ $(i \neq k)$. There follows $s(\varnothing) = 0$. By the Gleason theorem (Gleason, 1957) there exists an affine isomorphism $\rho \to s_\rho$ of the density operators (positive trace 1 operators) $K(H)$ onto S such that $s_\rho(x) = \mathrm{Tr}(P^x \rho)$, for every $\rho \in K(H)$, $x \in L$, where P^x is the orthogonal projection whose range is x. Then (L, S) is a pss (Gorini and Zecca, 1975).

Definition 3. (Berzi and Zecca, 1974). Let (L, S) be a pss. Then:

(i) $s \in S$ is a *maximal state* if $L(s)$ is a maximal dual principal ideal: $\bigwedge L(s)$ is then an atom of L. We denote by S_m the set of the maximal states of S.

(ii) $s \in S$ is a *characteristic state* if $s' \in S$ and $L(s) = L(s') \Rightarrow s = s'$. The set of the characteristic states is denoted by S_c.

At the level of a general pss one can show that $S_c \subset S_p \cap S_m$, with S_p being the set of the pure states (Berzi and Zecca, 1974). In the case of distributive logics, of the normal states of a W^*-algebra, and hence in the case of Remark 2, one can show that $S_c \equiv S_p \equiv S_m$ (Zecca, 1981*a*).

3. DEFINITION AND PROPERTIES OF INVARIANCE OF THE PRODUCT

The notion of a product of logics previously proposed by the author (Zecca, 1978, 1981*a*) is extended to include the states in order to define a product of proposition–state structures.

Definition 4. A proposition–state structure $(\mathbf{L}, \mathbf{S})$ is a *pseudoproduct* of the pss's (L, S) and $(\tilde{L}, \tilde{S})$ if there is a map $\bigcirc: L \times \tilde{L} \to \mathbf{L}$ (we write $\bigcirc(x, \tilde{x}) = x \circ \tilde{x}$) such that the following conditions hold:

P1. $\mathbf{1} \circ \tilde{\mathbf{1}} = \mathbf{1}_{\mathbf{L}}$, $x \circ \tilde{\varnothing} = \varnothing \circ \tilde{x} = \varnothing_{\mathbf{L}}$ for every $x \in L, \tilde{x} \in \tilde{L}$.

P2. $x \circ \tilde{x} = y \circ \tilde{y} \Leftrightarrow x = y, \tilde{x} = \tilde{y}$ $(x, y \in L, \tilde{x}, \tilde{y} \in L, x \neq \varnothing, y \neq \varnothing, \tilde{y} \neq \tilde{\varnothing}, \tilde{x} \neq \tilde{\varnothing})$.

P3. $A(L) \bigcirc A(\tilde{L}) \subset A(\mathbf{L})$, $A(L)$ being the atoms of L.

P4. $e(x) \circ e(\tilde{x}) = e(x \circ \tilde{x})$ for every $x \in L, \tilde{x} \in \tilde{L}$, where $e(x)$ denotes the central cover of x (Maeda and Maeda, 1970).

If $D \subset S, \tilde{D} \subset \tilde{S}$, denote

$$D \otimes \tilde{D} = \{s\tilde{s} | s \in D, \tilde{s} \in \tilde{D}; s\tilde{s}: L \times \tilde{L} \to [0, 1], (s\tilde{s})(x \circ \tilde{x}) = s(x)\tilde{s}(x)\}$$

Then we assume that $S \otimes \tilde{S}$ consists of the restriction to $L \times \tilde{L}$ of elements of $\mathbf{S}$ (which we denote again by $S \otimes \tilde{S}$) such that

P5. $S_1(x \circ \tilde{x}) = \overline{S_1(x) \otimes S_1(\tilde{x})}$ for every $x \in L, \tilde{x} \in \tilde{L}$.

P6. $S_0(x \circ \tilde{\mathbf{1}}) = \overline{S_0(x) \otimes \tilde{S}}$, $S_0(\mathbf{1} \circ \tilde{x}) = \overline{S \otimes S_0(\tilde{x})}$ for every $x \in L, \tilde{x} \in \tilde{L}$.

Proposition 1. Let the pss $(\mathbf{L}, \mathbf{S})$ be a pseudoproduct of (L, S) and $(\tilde{L}, \tilde{S})$. Then:

(i) $(\bigwedge_\alpha x_\alpha) \circ (\bigwedge_\beta \tilde{x}_\beta) = \bigwedge_{\alpha,\beta} (x_\alpha \circ \tilde{x}_\beta)$ for every $\{x_\alpha\} \subset L$, $\{\tilde{x}_\beta \subset \tilde{L}\}$.

(ii) $x \circ \tilde{x} \neq \varnothing_{\mathbf{L}}$; then $x \circ \tilde{x} \leq y \circ \tilde{y} \Leftrightarrow x \leq y, \tilde{x} \leq \tilde{y}$.

Proof. (i) From A3, P5, Lemma 1, and properties of set-theoretic intersection,

$$\begin{aligned} S_1(\textstyle\bigwedge_\alpha x_\alpha \circ \bigwedge_\beta y_\beta) &= \overline{\textstyle\bigcap_\alpha S_1(x_\alpha) \otimes \bigcap_\beta S_1(\tilde{x}_\beta)} \\ &= \textstyle\bigcap_{\alpha,\beta} \overline{S_1(x_\alpha) \otimes S_1(\tilde{x}_\beta)} \\ &= S_1(\textstyle\bigwedge_{\alpha,\beta} (x_\alpha \circ \tilde{x}_\beta)) \end{aligned}$$

(iii) $x \circ \tilde{x} \le y \circ \tilde{y}$ is equivalent to $S_1(x \circ \tilde{x}) \cap S_1(y \circ \tilde{y}) = S_1(x \circ \tilde{x})$. By (i) and A3, $S_1(x \wedge y \circ \tilde{x} \wedge \tilde{y}) = S_1(x \circ \tilde{x})$. By P2, $x \wedge y = x$ and $\tilde{x} \wedge \tilde{y} = \tilde{x}$ and hence $x \le y$, $\tilde{x} \le \tilde{y}$.

Proposition 2. Let the pss $(\mathbf{L}, \mathbf{S})$ be a pseudoproduct of (L, S) and $(\tilde{L}, \tilde{S})$. If $D \subset S$, $\tilde{D} \subset \tilde{S}$, $s \in S$, $\tilde{s} \in \tilde{S}$, then the following conditions are equivalent:

(i) $\bigwedge L(D \otimes \tilde{D}) = \bigwedge L(D) \circ \bigwedge L(\tilde{D})$.
(ii) $\overline{\bar{D} \otimes \bar{\tilde{D}}} \equiv \overline{D \otimes \tilde{D}}$.
(iii) $L(s) \supset L(D)$ and $L(\tilde{s}) \supset L(\tilde{D}) \Rightarrow L(s\tilde{s}) \supset L(D \otimes \tilde{D})$.

Proof. (i) $\Leftrightarrow$ (ii) from P5,

$$D \otimes \tilde{D} = S_1(\bigwedge L(D \otimes \tilde{D})) = \overline{S_1(\bigwedge L(D)) \otimes S_1(\bigwedge L(\tilde{D}))} = \overline{\bar{D} \otimes \bar{\tilde{D}}}$$

(ii) $\Rightarrow$ (iii). From $s \in D, \tilde{s} \in \tilde{D}$,

$$s\tilde{s} \in \bar{D} \otimes \bar{\tilde{D}} \subset \overline{\bar{D} \otimes \bar{\tilde{D}}} = \overline{D \otimes \tilde{D}}$$

Hence $L(s\tilde{s}) \supset L(D \otimes \tilde{D})$.

(iii) $\Rightarrow$ (ii). We have

$$\overline{\bar{D} \otimes \bar{\tilde{D}}} \supset \overline{D \otimes \tilde{D}}$$

from a property of closure under superposition. On the other hand

$$s \in \bar{D}, \qquad \tilde{s} \in \bar{\tilde{D}}$$

that is,

$$s\tilde{s} \in \bar{D} \otimes \bar{\tilde{D}}$$

implies, by the assumptions,

$$s\tilde{s} \in \overline{D \otimes \tilde{D}}$$

Hence

$$\overline{\bar{D} \otimes \bar{\tilde{D}}} \subset \overline{D \otimes \tilde{D}}, \qquad \overline{D \otimes \tilde{D}} \quad \text{being closed}$$

Definition 5. A proposition–state structure $(\mathbf{L}, \mathbf{S})$ is a *product* of the pss's (L, S) and $(\tilde{L}, \tilde{S})$ if it is a pseudoproduct of them and any one of the equivalent conditions of Proposition 2 is satisfied. In the case of the product we write $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$.

Proposition 3. Let $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$. Then:

(i) $\bigvee_{\alpha,\beta} (x_\alpha \circ x_\beta) = \bigvee_\alpha x_\alpha \circ \bigvee_\beta x_\beta$, $\{x_\alpha\} \subset L$, $\{\tilde{x}_\beta\} \subset \tilde{L}$.
(ii) $(x \circ \tilde{x})^\perp = (x^\perp \circ \tilde{x}^\perp) \vee (x^\perp \circ \tilde{x}) \vee (x \circ \tilde{x}^\perp)$, $x \in L$, $\tilde{x} \in \tilde{L}$.

Proof. (i). One has from the very definition of the product of states and the property of the closure under superposition

$$\bigcup_\beta \overline{S_1(x) \otimes S_1(\tilde{x}_\beta)} \supset S_1(x) \otimes \bigcup_\beta S_1(\tilde{x}_\beta)$$

By taking the closure, from P5 and Proposition 2(ii), we have

$$S_1(\bigvee_\beta (x \circ \tilde{x}_\beta)) \supset \overline{\overline{S_1(x)} \otimes \overline{\bigcup_\beta S_1(\tilde{x}_\beta)}} = \overline{S_1(x) \otimes S_1(\bigvee_\beta \tilde{x}_\beta)} = S_1(x \circ \bigvee_\beta \tilde{x}_\beta)$$

Therefore $\bigvee_\beta (x \circ \tilde{x}_\beta) \geq x \circ \bigvee_\beta \tilde{x}_\beta$. But $\bigvee_\beta (x \circ \tilde{x}_\beta) \leq x \circ \bigvee_\beta \tilde{x}_\beta)$ as a consequence of Proposition 1(ii). Therefore $x \circ \bigvee_\beta \tilde{x}_\beta = \bigvee_\beta (x \circ \tilde{x}_\beta)$. The distributivity of $\bigvee$ over the other component of the product can be shown in a similar way.

(ii) One has by P5, P6, Proposition 1, and (i):

$$\begin{aligned}(x \circ \tilde{x})^\perp &= (x \circ \mathbf{1})^\perp \vee (\mathbf{1} \circ \tilde{x})^\perp \\ &= (x^\perp \circ \tilde{\mathbf{1}}) \vee (\tilde{\mathbf{1}} \circ \tilde{x}^\perp) \\ &= [x^\perp \circ (\tilde{x} \vee \tilde{x}^\perp)] \vee [x \vee x^\perp \circ \tilde{x}^\perp] \\ &= x^\perp \circ \tilde{x} \vee x^\perp \circ \tilde{x}^\perp \vee x \circ \tilde{x}^\perp\end{aligned}$$

Besides the superposition also the maximality of the states is preserved under the product.

Lemma 2. Let $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$, $s \in S_m$, $\tilde{s} \in \tilde{S}_m$. Then $s\tilde{s} \in \mathbf{S}_m$.

Proof. From the assumptions, $\bigwedge L(s) \in A(L)$, $\bigwedge L(\tilde{s}) \in A(\tilde{L})$. By P3, $\bigwedge L(s) \circ \bigwedge L(\tilde{s}) \in A(\mathbf{L})$ and since $L(s\tilde{s}) \supset \{x \circ \tilde{x}: x \in L(s), \tilde{x} \in L(\tilde{s})\}$, it follows that $\bigwedge L(s\tilde{s}) \leq \bigwedge L(s) \circ L(\tilde{s})$. Hence $\bigwedge L(s\tilde{s}) \in A(\mathbf{L})$ and $L(s\tilde{s})$ is a maximal dual principal ideal.

As a consequence, an alternative formulation of P3 could be that of requiring that the products of maximal states give maximal states.

Another consequence is that the product of characteristic states is in general a maximal state. In order that $s\tilde{s}$ be characteristic in $\mathbf{S}$ when both s and $\tilde{s}$ are characteristic in S, $\tilde{S}$, it is necessary and sufficient that

$$\{s\} \otimes \{\tilde{s}\} = \overline{\{s\} \otimes \{\tilde{s}\}}$$

Indeed $s\tilde{s}$ is characteristic iff $\{s\tilde{s}\} \equiv S_1(\bigwedge L(s\tilde{s}))$, hence iff

$$\{s\tilde{s}\} = \overline{\{s\tilde{s}\}}$$

But $\{s\tilde{s}\} = \{s\} \otimes \{\tilde{s}\}$. This is the case, for instance, of the normal states of a W^*-algebra (Zecca, 1981*b*).

4. GENERAL RESULTS AND SPECIAL EXAMPLES

Suppose now $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$. Then:

(a) The maps $\mu: L \to \mathbf{L}$ and $\tilde{\mu}: \tilde{L} \to \mathbf{L}$ defined by $\mu(x) = x \circ \tilde{\mathbf{1}}$, $\tilde{\mu}(\tilde{x}) = \mathbf{1} \circ \tilde{x}$ are unitary c-morphisms in the sense of Aerts (1984), as a consequence of the previous results, such that:

(b) $\mu(x)$ commutes with $\tilde{\mu}(\tilde{x})$ for every $x \in L, \tilde{x} \in \tilde{L}$, since $(\mathbf{1} \circ \tilde{x} \wedge x \circ \tilde{\mathbf{1}}) \vee (\mathbf{1} \circ \tilde{x} \wedge \tilde{x}^{\perp} \circ \tilde{\mathbf{1}}) = \mathbf{1} \circ \tilde{x}$, as one can check by using the results of the previous section.

(c) Moreover, $\mu(e) \wedge \tilde{\mu}(\tilde{e}) = e \circ \tilde{e}$ is an atom of $\mathbf{L}$ whenever $e \in A(L)$, $\tilde{e} \in A(\tilde{L})$. Hence $(\mu, \tilde{\mu}, \mathbf{L})$ is a solution of the coupling conditions for logics in the sense of Aerts (1984).

(d) If $e \circ \tilde{e} \leq f \circ \tilde{f} \vee g \circ \tilde{g}$ with $f \neq g, \tilde{f} \neq \tilde{g}$, $e, f, g \in A(L)$, $\tilde{e}, \tilde{f}, \tilde{g} \in A(\tilde{L})$, then $e \circ \tilde{e} = f \circ \tilde{f}$ or $e \circ \tilde{e} = g \circ \tilde{g}$. Indeed by using the fact that a triple of elements of an orthomodular lattice is distributive if some one of them commutes with the other two (Maeda and Maeda, 1979), from Proposition 3 we have

$$e \circ \tilde{e} = e \circ \tilde{e} \wedge \mathbf{1} \circ \tilde{e} \leq [f \circ \tilde{\mathbf{1}} \vee \mathbf{1} \circ \tilde{g}] \wedge \mathbf{1} \circ \tilde{e} = f \circ \tilde{e} \vee \mathbf{1} \circ (\tilde{e} \wedge \tilde{g}) \leq f \circ \tilde{e}$$

so that $e = f$. Analogously, $\tilde{e} = \tilde{f}$. Hence the product atoms $g \circ \tilde{g}, f \circ \tilde{f}$ with $g \neq f$ and $f \neq \tilde{g}$ are separated by a superselection rule (Piron, 1976). An analogous property holds in the schemes of Aerts (1985) and Pulmannová (1985).

(e) If $A(L) \bigcirc A(\tilde{L}) = A(\mathbf{L})$, an immediate consequence of point (d) is that

$$[e \circ \tilde{e} \vee f \circ \tilde{f}] \wedge (f \circ \tilde{f})^{\perp} \equiv e \circ \tilde{e} \wedge (f \circ \tilde{f})^{\perp}$$

for every $e, f \in A(L)$, $\tilde{e}, \tilde{f} \in A(\tilde{L})$ with $e \neq f, \tilde{f} \neq \tilde{e}$ [compare with the results in Pulmannová (1985)]. If now $e \circ \tilde{e} \wedge (f \circ \tilde{f})^{\perp} = \varnothing_{\mathbf{L}}$, then from the orthomodularity of $\mathbf{L}$ we would have the contradiction

$$e \circ \tilde{e} \vee f \circ \tilde{f} = f \circ \tilde{f} \vee [(f \circ \tilde{f})^{\perp} \wedge (e \circ \tilde{e} \vee f \circ \tilde{f})] = f \circ \tilde{f}$$

Hence $e \circ \tilde{e} \perp f \circ \tilde{f}$ for every $e, f \in A(L)$, $\tilde{e}, \tilde{f} \in A(\tilde{L})$, $e \neq f$, $\tilde{e} \neq \tilde{f}$, since $e \circ \tilde{e}$ is an atom.

Proposition 4. Let $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$ and $A(L) \bigcirc A(\tilde{L}) = A(\mathbf{L})$. Then at least one of $L, \tilde{L}$ is a Boolean logic.

Proof. Let L be nondistributive and choose $\tilde{e}, \tilde{f} \in A(\tilde{L})$ such that $\tilde{e} \wedge \tilde{f}^{\perp} = \tilde{\varnothing}$, $e, f \in A(L)$, $e \neq f$. Since $e \circ \tilde{e} \perp f \circ \tilde{f}$ as previously pointed out, by taking into account Propositions 1 and 3 and distributivity

$$\begin{aligned} e \circ \tilde{e} &= e \circ \tilde{e} \wedge \mathbf{1} \circ \tilde{e} \leq [f^{\perp} \circ \tilde{\mathbf{1}} \vee \mathbf{1} \circ \tilde{f}^{\perp}] \wedge \mathbf{1} \circ \tilde{e} \\ &= f^{\perp} \circ \tilde{e} \vee (\mathbf{1} \circ (\tilde{f}_{\perp} \wedge \tilde{e})) = f^{\perp} \circ \tilde{e} \end{aligned}$$

and hence $e \leq f^{\perp}$ by Proposition 1(iii). The atoms of L are then mutually orthogonal and hence L is distributive.

Results similar to those of Proposition 4 have been found by Aerts (1982) and Pulmannová (1983, 1985) in their schemes. In general we have the following results.

Lemma 3. Let L be any logic. Then $A(C(L)) \equiv \{e(x): x \in A(L)\}$.

Proof. See Zecca (1981) or Beltrametti and Cassinelli (1981).

Lemma 4. Let $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$. Then:
(i) $A(C(L)) \bigcirc A(C(\tilde{L})) = A(C(\mathbf{L}))$.
(ii) $C(\mathbf{L})$ is the distributive logic generated by $C(L) \bigcirc C(\tilde{L})$.

Proof. (i) By Lemma 3 and P3, if $a \in A(C(L)), \tilde{a} \in A(C(\tilde{L}))$, then $a \circ \tilde{a} = e(x) \circ e(\tilde{x}) = e(x \circ \tilde{x}) \in A(C(\mathbf{L}))$ for some $x \in A(L), \tilde{x} \in A(\tilde{L})$. Conversely, if $A \in A(C(\mathbf{L})), A \notin A(C(L)) \bigcirc A(C(\tilde{L}))$ we have the contradiction

$$\begin{aligned} A = A \wedge \mathbf{1} \circ \tilde{\mathbf{1}} &= A \wedge [(\textstyle\bigvee_\alpha e(x_\alpha)) \circ (\bigvee_\beta e(\tilde{x}_\beta))] \\ &= A \wedge [\textstyle\bigvee_{\alpha,\beta} (e(x_\alpha) \circ e(\tilde{x}_\beta))] \\ &= \textstyle\bigvee_{\alpha,\beta} [A \wedge (e(x_\alpha) \circ e(\tilde{x}_\beta))] \\ &= \varnothing_{\mathbf{L}} \end{aligned}$$

where we have used the fact that A is a central element of a complete lattice and where we have set $\{x_\alpha\} = A(L), \{\tilde{x}_\beta\} = A(\tilde{L})$.

(ii) If $a = \bigvee_a e(x_\alpha), \{x_\alpha\} \subset A(L)$ and $\tilde{a} = \bigvee_\beta e(\tilde{x}_\beta), \{\tilde{x}_\beta) \subset A(\tilde{L})$, then

$$a \circ \tilde{a} = \textstyle\bigvee_\alpha e(x_\alpha) \circ \bigvee_\beta e(\tilde{x}_\beta) = \bigvee_{\alpha,\beta} (e(x_\alpha) \circ e(\tilde{x}_\beta)) = \bigvee_{\alpha,\beta} (e(x_\alpha) \circ e(x_\beta))$$

is an element of $C(\mathbf{L})$ by the point (i). On the other hand, if $X \in C(\mathbf{L})$, then $X = \bigvee_\alpha A_\alpha$ for some $\{A_\alpha\} \subset A(C(\mathbf{L}))$. By Lemma 3 and point (i), $A = \bigvee_\alpha e(x_\alpha \circ \tilde{x}_\alpha) = \bigvee_\alpha (e(x_\alpha) \circ e(\tilde{x}_\alpha))$. [Compare with Zecca (1981).]

We have also the following characterization.

Lemma 5. Let $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$. Then the following conditions are equivalent:

(i) $\mathbf{L}$ is a distributive logic.
(ii) $L, \tilde{L}$ are distributive logics and $A(L) \bigcirc A(\tilde{L}) = A(\mathbf{L})$.

Proof. If $\mathbf{L}$ is distributive, so are L and $\tilde{L}$, since, as previously pointed out in (a), μ and $\tilde{\mu}$ are orthoisomorphisms from L and $\tilde{L}$ into $\mathbf{L}$. From

$C(L) = L$, $C(\tilde{L}) = \tilde{L}$, $C(\mathbf{L}) = \mathbf{L}$, and Lemma 4(i), we have then $A(L) \bigcirc A(\tilde{L}) = A(\mathbf{L})$.

Suppose now condition (ii) holds. Let $X \in \mathbf{L}$ and $X = \bigvee_\alpha A_\alpha$ for some $\{A_\alpha\} \subset A(\mathbf{L})$. From Lemma 4(i) and the assumption it follows that $A(\mathbf{L}) = A(C(\mathbf{L}))$. Hence $X \in C(\mathbf{L})$ or $C(\mathbf{L}) = \mathbf{L}$.

Lemma 6. Let $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$. Then $\tilde{L}$ is an irreducible logic if and only if both $L, \tilde{L}$ are irreducible logics.

Proof. If $C(\mathbf{L}) = \{\varnothing_\mathbf{L}, \mathbf{1}_L\}$, from Lemma 4(i), one must have $C(L) = \{\varnothing, \mathbf{1}\}$ and $C(\tilde{L}) = \{\varnothing, \tilde{\mathbf{1}}\}$. Conversely, if $L, \tilde{L}$ are irreducible, then from P1, $C(L) \bigcirc C(\tilde{L}) = \{\varnothing_\mathbf{L}, \mathbf{1}_L\}$, which is itself a distributive logic. From Lemma 4(ii) it follows that $C(\mathbf{L}) = \{\varnothing_\mathbf{L}, \mathbf{1}_\mathbf{L}\}$.

According to the above results, we have the following situations.

Remark 3. Suppose now the physical systems Σ and $\tilde{\Sigma}$ interact, $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S}\}$, in such a way that $\Sigma + \tilde{\Sigma}$ is a classical system. The logic $\mathbf{L}$ is then distributive, so that it can be identified with the power set of its atoms: $\mathbf{L} \cong P(A(\mathbf{L}))$. By Lemma 4 both the interacting systems Σ and $\tilde{\Sigma}$ are classical systems, and $\mathbf{L} \cong P(A(\mathbf{L})) = P(A(L) \times A(\tilde{L}))$ (by using also P2). If we identify S with the atoms (pure states) of L and $\tilde{S}$ with atoms of $\tilde{L}$, then

$$\mathbf{S} \equiv \overline{S \otimes \tilde{S}} = S \otimes \tilde{S}$$

that is, roughly speaking, the states of the product are the product of the states. This is coherent also with a distinguishing aspect of classical physical systems, namely that for distributive logics the pure states do not admit purely quantum superpositions (see, for instance, Jauch, 1968; Beltrametti and Cassinelli, 1981; Zecca, 1981*b*).

Remark 4. Suppose with the notations of Remark 2 and Definition 4, $(L(\mathbf{H}), \mathbf{S}) = (L(H) \otimes L(\tilde{H}), K(H) \otimes K(\tilde{H}))$, $H, \tilde{H}, \mathbf{H}$ being separable Hilbert spaces with $\dim(H) \geq 3$, $\dim(\tilde{H}) \geq 3$, $\mathbf{S} \subset K(\mathbf{H})$. Since, as mentioned, our product of logics satisfies the coupling conditions given by Aerts (1984), there are only two inequivalent solutions of the product: $L(\mathbf{H}) \cong L(H \otimes \tilde{H})$ or $L(\mathbf{H}) \cong L(H^* \otimes \tilde{H})$ [H^* being the dual of H (Aerts and Daubachies, 1978)]. The two solutions have to be considered physically inequivalent since, as stressed by Aerts (1984), none of them is a solution of the universal problem for the coupling of logics.

For the states, all is well because in the Hilbert model

$$S_1(a \otimes \tilde{a}) = \overline{S_1(a) \otimes S_1(\tilde{a})} \qquad \text{for every} \quad a \in L(H), \quad \tilde{a} \in L(\tilde{H})$$

and the superposition of the (trace class) states is preserved under the Hilbert tensor product, as shown in Zecca (1981*b*, Section 3.3). Hence

$$\mathbf{S} = \overline{S_1(\mathbf{1}) \otimes S_1(\tilde{\mathbf{1}})} \equiv \overline{K(H) \otimes K(\tilde{H})} = K(H \otimes \tilde{H})$$

There are then two solutions to our problem of the product of the pss in the Hilbert model: $(L(H \otimes \tilde{H}), K(H \otimes \tilde{H}))$ and $(L(H^* \otimes \tilde{H}), K(H^* \otimes \tilde{H}))$.

5. THE INTERACTION OF A CLASSICAL AND A PURELY QUANTUM SYSTEM

This is the case of interest in connection with the problem of measurement of a quantum system by means of a classical apparatus (Ludwig, 1973; Hellwig and Krausser, 1977; Piron, 1976).

Proposition 5. Let $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$, $C(L) = L$, $C(\tilde{L}) = \{\tilde{\varnothing}, \tilde{\mathbf{1}}\}$. Then the following conditions are equivalent:

(i) $A(L) \bigcirc A(\tilde{L}) = A(\mathbf{L})$.
(ii) $L[\varnothing_{\mathbf{L}}, x \circ \tilde{\mathbf{1}}] = x \circ \tilde{L}$ for every $x \in A(L)$.

Proof. (i) $\Rightarrow$ (ii). From Proposition 1(iii) one has in general $L[\varnothing_{\mathbf{L}}, x \circ \tilde{\mathbf{1}}] \supset x \circ \tilde{L}$. On the other hand, if a is an atom of $L[\varnothing_{\mathbf{L}}, x \circ \tilde{\mathbf{1}}]$, then also $a \in A(\mathbf{L})$ (Maeda and Maeda, 1970), so that, by (i), $a = x' \circ \tilde{x} \leq x \circ \tilde{\mathbf{1}}$ for some $x' \in A(L), \tilde{x} \in A(\tilde{L})$. Hence from Proposition 1(iii), $a = x \circ \tilde{x}$. If now $A \in L[\varnothing_{\mathbf{L}}, x \circ \tilde{\mathbf{1}}]$, then $A = \bigvee_\alpha \alpha_\alpha$, $\{a_\alpha\}$ being atoms of $L[\varnothing_{\mathbf{L}}, x \circ \tilde{\mathbf{1}}]$. By the previous result and Proposition 3(i), $A = \bigvee_\alpha (x \circ x_\alpha) = x \circ \bigvee_\alpha x_\alpha \in x \circ \tilde{L}$.

(ii) $\Rightarrow$ (i) Let $A \in A(\mathbf{L})$. By setting $\{x_\alpha\} \equiv A(L)$, we have $A = A \wedge (\bigvee_\alpha x_\alpha \circ \tilde{\mathbf{1}}) = A \wedge [(\bigvee_\alpha (x_\alpha \circ \tilde{\mathbf{1}})] = \bigvee_\alpha [A \wedge (x_\alpha \circ \tilde{\mathbf{1}})]$, since, from Lemma 3, $A(L) \bigcirc \tilde{\mathbf{1}} \equiv A(C(\mathbf{L}))$. Hence $A \leq x \circ \tilde{\mathbf{1}}$ for some $x \in A(L)$. Therefore $A = x \circ \tilde{x}, \tilde{x} \in A(\tilde{L})$ by the assumption (ii).

If now $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$ with $C(L) = L$, $C(\tilde{L}) = \{\tilde{\varnothing}, \tilde{\mathbf{1}}\}$ and any one of the conditions of Proposition 4 is satisfied, by standard results in lattice decomposition theory

$$\mathbf{L} = \bigoplus (x \circ \tilde{L} \colon x \in A(L))$$

that is, the compound system can be interpreted as a quantum system endowed with continuous superselection rules labeled by the points of the classical system, the mutually orthogonal sectors being replicas of the irreducible quantum system. This interpretation is supported also by the fact that $\mathbf{L}$ is defined up to an orthoisomorphism, as can be directly shown [compare with Zecca (1978); for analogous results in similar schemes see Aerts (1984) and Pulmannová (1983, 1985)]. Indeed if $a \in \mathbf{L}$, from the uniqueness of the decomposition $a = \bigoplus_\alpha a_\alpha$ and the result of Proposition 5, one has

$$a_\alpha = a \wedge x_\alpha \circ \tilde{1} = x_\alpha \circ \tilde{x}_\alpha(a) \qquad (\{x_\alpha\} \equiv A(L))$$

and also

$$(\bigvee_k a_k)_\alpha = x_\alpha \circ \tilde{x}_\alpha(\bigvee_k a_k)$$

$$(\bigwedge_k a_k)_\alpha = x_\alpha \circ \tilde{x}_\alpha(\bigwedge_k a_k)$$

$$(a^\perp)_\alpha = x_\alpha \circ \tilde{x}_\alpha^\perp(a)$$

for every $\{a_\alpha\} \subset \mathbf{L}$. If now $(\mathbf{L}', \mathbf{S}')$ is another product of (L, S) and $(\tilde{L}, \tilde{S})$ with $C(L) = L$, $C(\tilde{L}) = \{\tilde{\varnothing}, \tilde{1}\}$ and any one of the equivalent conditions of Proposition 5 is satisfied, then $\mathbf{L}', \mathbf{L}$ come out to be orthoisomorphic through the map

$$a = \bigvee_\alpha (x_\alpha \circ \tilde{x}_\alpha(a)) \to a' = \bigvee\nolimits'_\alpha (x_\alpha \circ' \tilde{x}_\alpha(a))$$

For the states, we are now in a position to show that the product states are a solution for the states of the product. If L is distributive and $x \in A(L)$, define $s_x: L \to [0, 1]$ by $s_x(a) = 1$ if $x \le a$, $s_x(a) = 0$ if $x \wedge a = \varnothing$. If $\tilde{m} \in \tilde{S}$, define $s_x\tilde{m}: \mathbf{L} \to [0, 1]$ by [compare with Pulmannová (1985)]

$$(s_x\tilde{m})(a) = (s_x\tilde{m})(\bigoplus(y \circ \tilde{y}(a): y \in A(L))) = \tilde{m}(\tilde{x}(a)) \qquad (a \in \mathbf{L})$$

Lemma 7. Let $(\mathbf{L}, \mathbf{S}) = (L \otimes \tilde{L}, S \otimes \tilde{S})$, $C(L) = L$, $C(\tilde{L}) = \{\tilde{\varnothing}, \tilde{1}\}$, and any one of the conditions of Proposition 5 be satisfied. Then $\mathbf{S} = \{s_x\tilde{m}: x \in A(L), \tilde{m} \in \tilde{\mathbf{S}}\}$ is a minimal solution for the states of the product in the sense that

$$S_1(a) \otimes S_1(\tilde{a}) = \overline{S_1(a) \otimes S_1(\tilde{a})} \qquad \text{for every} \quad a \in L, \quad \tilde{a} \in \tilde{L}$$

Proof. By setting $A(a) = \{y \in A(L): y \le a\}$ we have

$$\begin{aligned}(s_x\tilde{m})(a \circ \tilde{a}) &= (s_x\tilde{m})(\bigoplus(y \circ \tilde{a}: y \in A(a)))\\ &= (s_x\tilde{m})(\bigoplus(y \circ \tilde{x}(\tilde{a}): y \in A(a)))\\ &= \tilde{m}(\tilde{x}(\tilde{a}))\end{aligned}$$

where $\tilde{x}(\tilde{a}) = \tilde{a}$ if $y \in A(a)$, $\tilde{x}(\tilde{a}) = \tilde{\varnothing}$ if $y \notin A(a)$. Then $s_x\tilde{m} \in S_1(a \circ \tilde{a})$ iff $x \le a$ and $\tilde{m}(\tilde{a}) = 1$ and hence $S_1(a \circ \tilde{a}) = S_1(a) \otimes S_1(\tilde{a})$.

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